

Center for

Particle Cosmology

at the University of Pennsylvania

Scale Invariance from Conformal Invariance

Justin Khoury (UPenn)

with K. Hinterbichler, A. Joyce and G. Miller,
to appear

Framework

- Non-inflationary mechanism

- Spontaneous breaking of conformal invariance:

$$SO(4, 2) \rightarrow SO(4, 1)$$

- Gravity is unimportant, can describe mechanism in flat space

- Related ideas

Rubakov's U(1) model
[Rubakov, 0906:3693](#)

Galilean Genesis

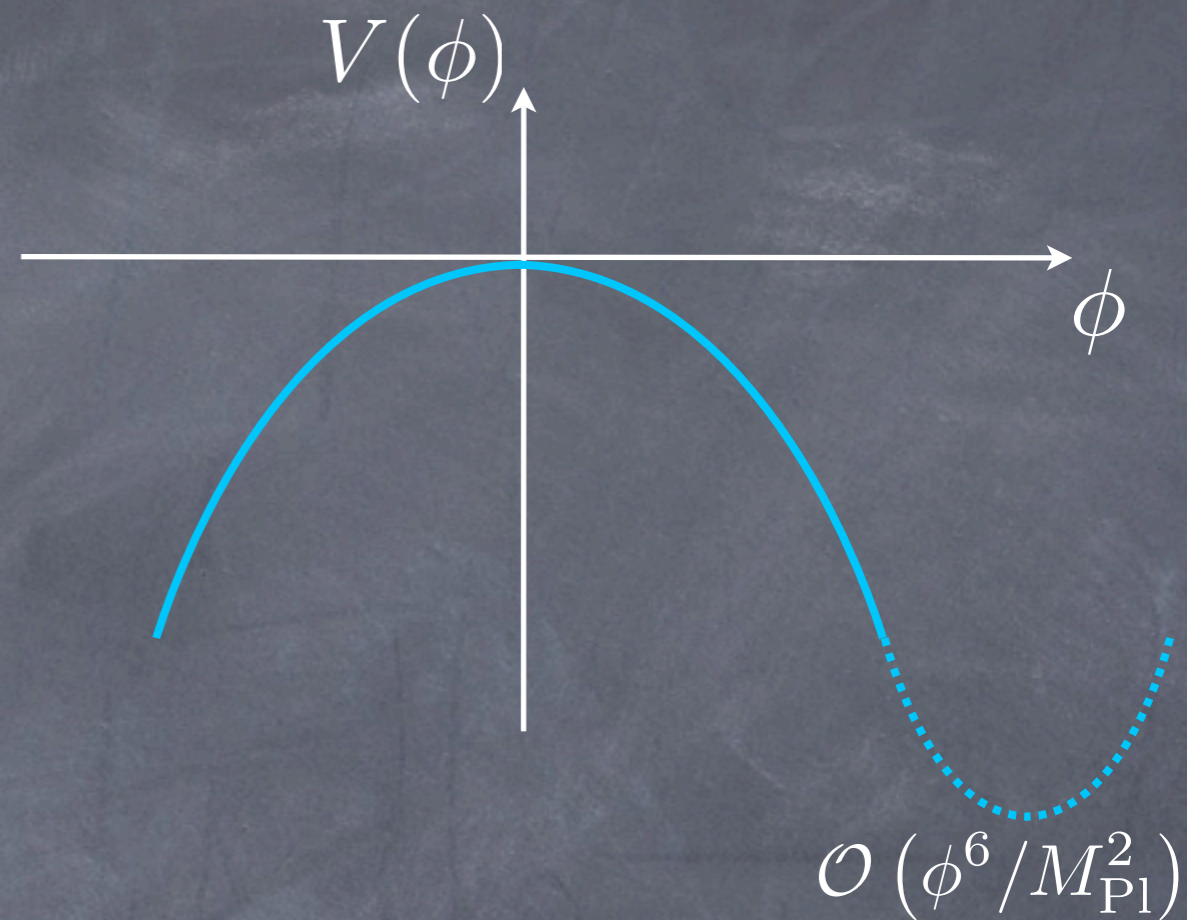
[Creminelli, Nicolis & Trincherini, 1007:0027](#)

- DBI realization

Flat space: simplest realization

$$S_\phi = \int d^4x \left(-\frac{1}{2}(\partial\phi)^2 + \frac{\lambda}{4}\phi^4 \right)$$

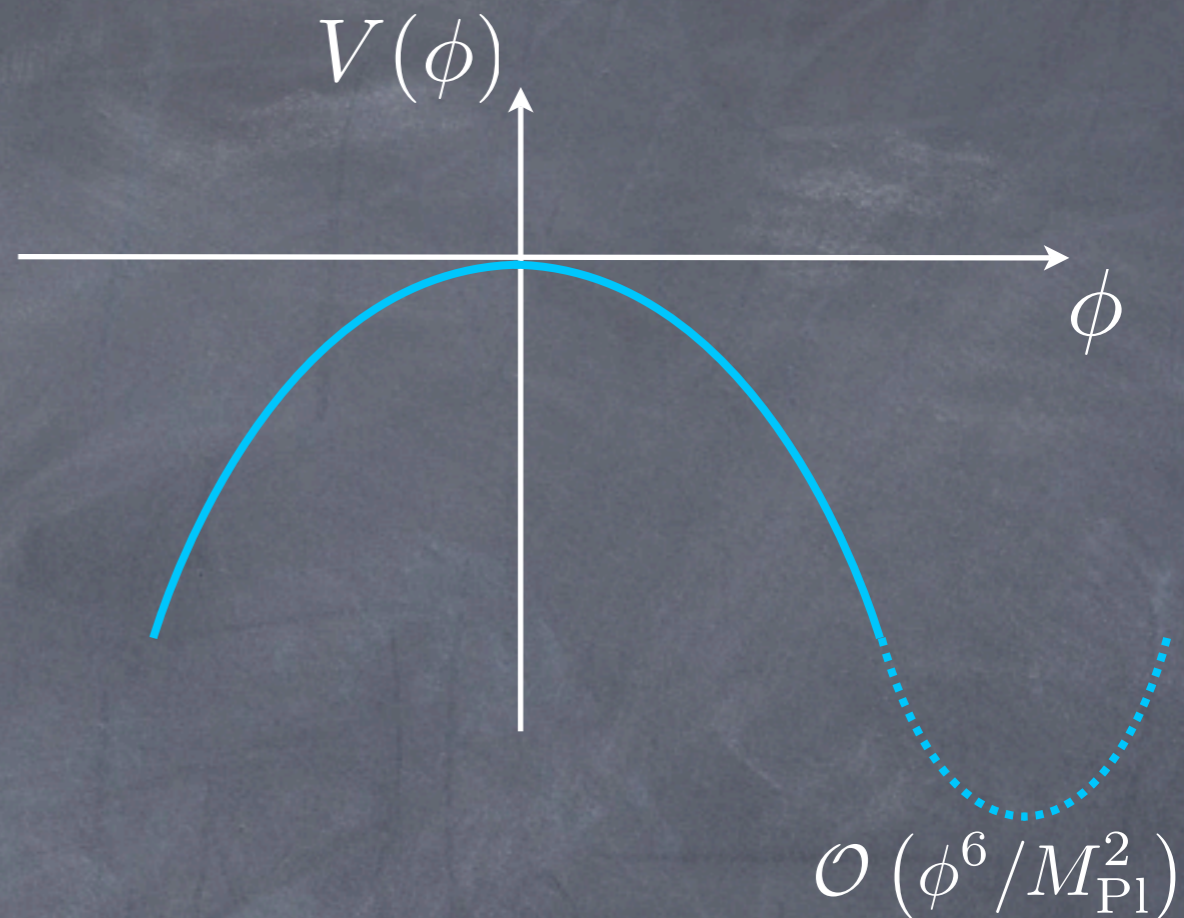
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$$\delta_{P_\mu} \phi = -\partial_\mu \phi;$$

$$\delta_{J^{\mu\nu}} \phi = (x^\mu \partial^\nu - x^\nu \partial^\mu) \phi;$$

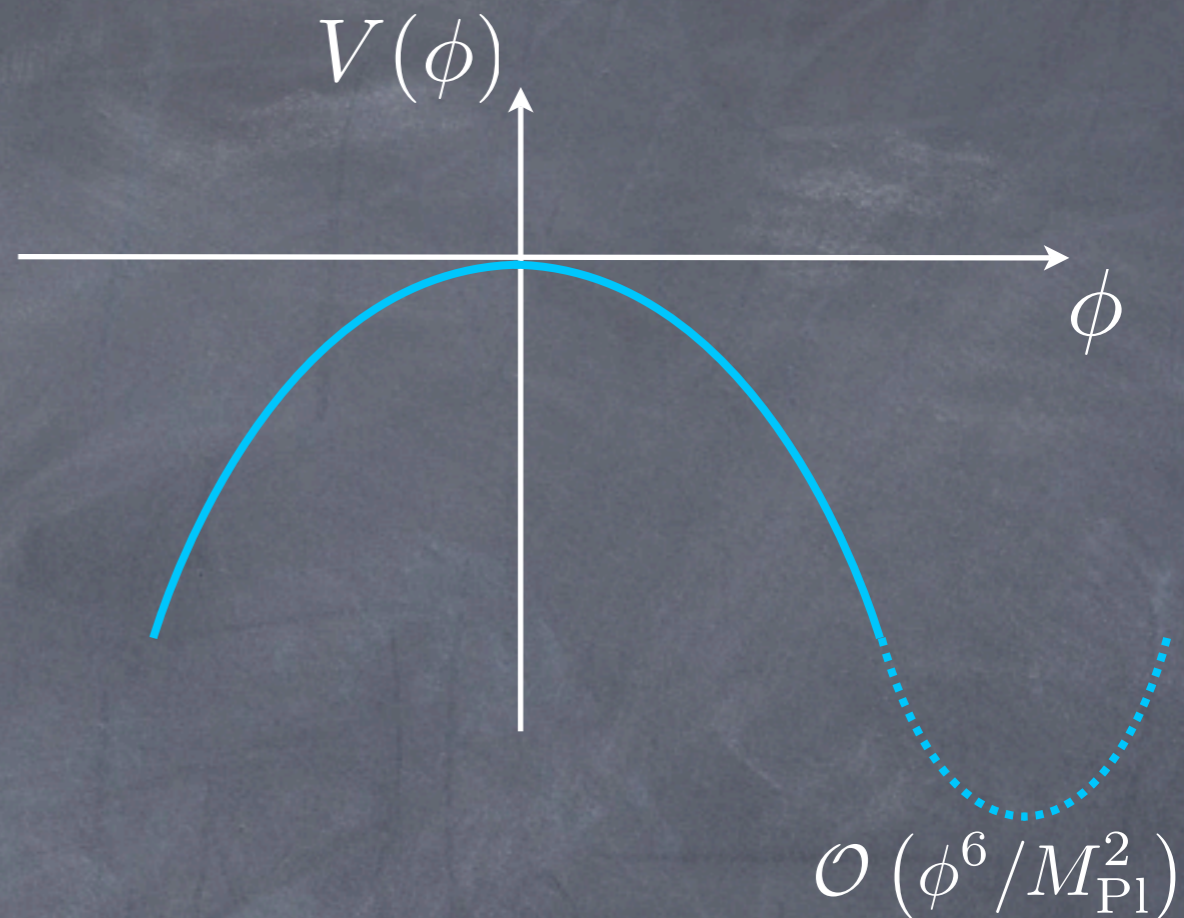
$$\delta_D \phi = -(1 + x^\mu \partial_\mu) \phi;$$

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Assume existence of conformal weight 0 fields χ

$$S_\chi = -\frac{1}{2} \int d^4x \left(\phi^2 \eta^{\mu\nu} \partial_\mu \chi \partial_\nu \chi + \phi^4 \lambda_\chi \chi^2 \right)$$

$\implies \chi$ couples to an effective metric $g_{\mu\nu}^{\text{eff}} = \phi^2 \eta_{\mu\nu}$

Spontaneous breaking

Assuming homogeneous evolution,

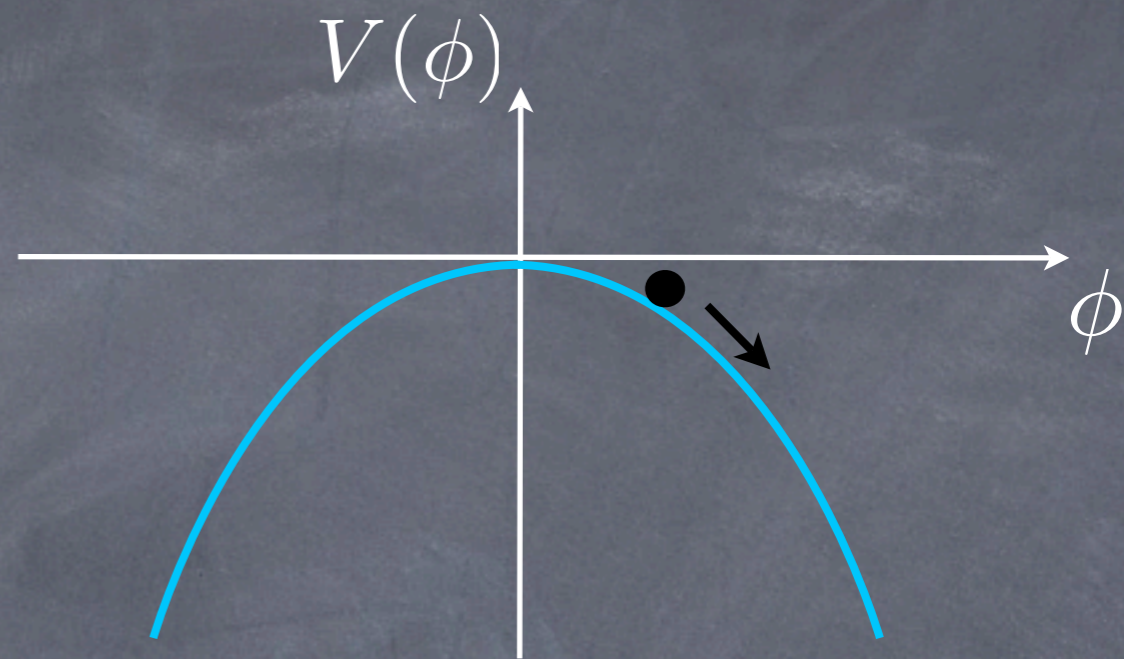
$$\ddot{\phi} = \lambda\phi^3$$

\implies

$$\phi(t) = \frac{\sqrt{2}}{\sqrt{\lambda(-t)}}$$

(assuming $E = 0$)

$$-\infty < t < 0$$



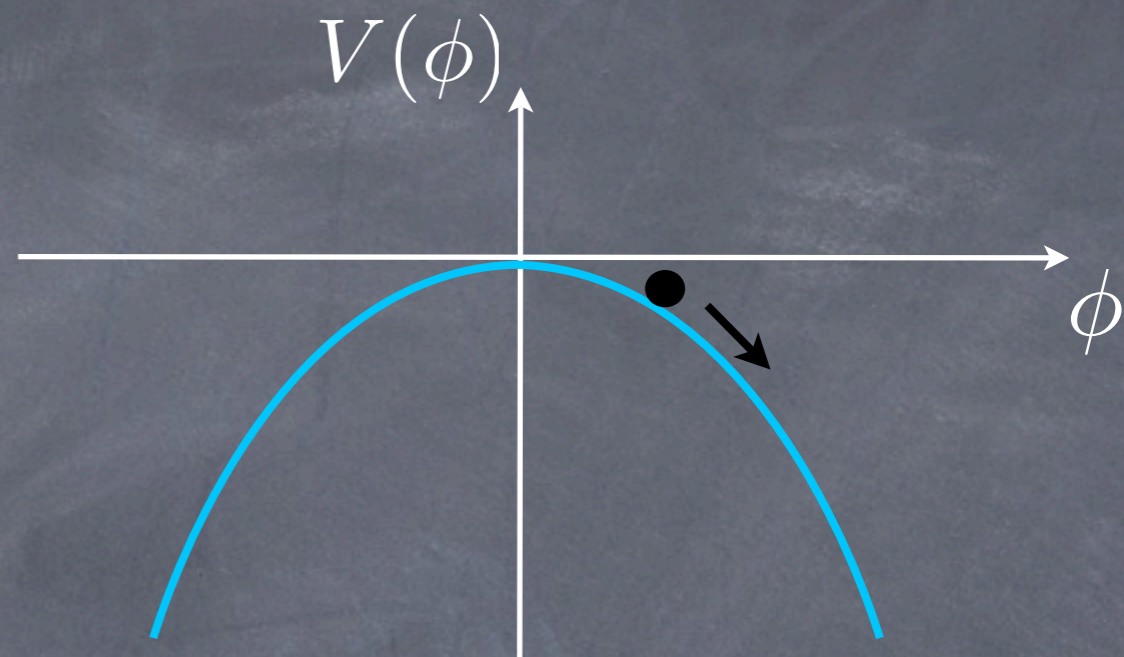
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Coming back to $S_\chi = -\frac{1}{2} \int d^4x (\phi^2 \eta^{\mu\nu} \partial_\mu \chi \partial_\nu \chi + \phi^4 \lambda_\chi \chi^2)$

$$\implies g_{\mu\nu}^{\text{eff}} = \phi^2 \eta_{\mu\nu} \sim \frac{1}{t^2} \eta_{\mu\nu} \quad \text{de Sitter}$$

$\therefore \chi$ will acquire scale inv. perturbations
(if λ_χ is sufficiently small)

Symmetry Breaking Pattern

Background $\bar{\phi}(t) \sim \frac{1}{t}$ is annihilated by the 10 generators:

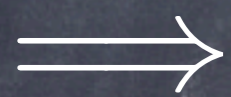
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Define: $\delta_{J^{-2,-1}} = \delta_D$, $\delta_{J^{-2,i}} = \frac{1}{2}(\delta_{P_i} - \delta_{K_i})$, $\delta_{J^{-1,i}} = \frac{1}{2}(\delta_{P_i} + \delta_{K_i})$.



$$[\delta_{J_{ab}}, \delta_{J_{cd}}] = \eta_{ac}\delta_{J_{bd}} - \eta_{bc}\delta_{J_{ad}} + \eta_{bd}\delta_{J_{ac}} - \eta_{ad}\delta_{J_{bc}}$$

Commutation relations of $SO(4,1)$

$$\eta_{ab} = \text{diag}(-1, 1, 1, 1, 1)$$

$$\therefore SO(4,2) \rightarrow SO(4,1)$$

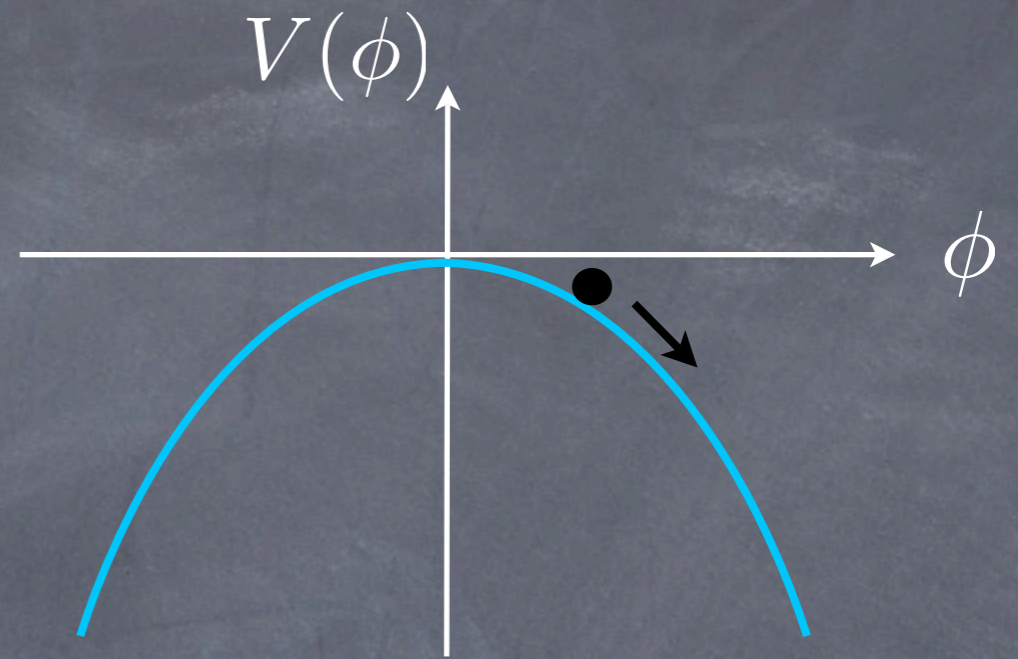
Perturbations: Attractor Property

$$\ddot{\varphi}_k + k^2 \varphi_k = 3\lambda \bar{\phi}^2 \varphi_k = \frac{6}{t^2} \varphi_k$$

$$\implies \varphi_k \sim \frac{1}{t^2} \quad \text{and} \quad \varphi_k \sim t^3$$

growing decaying

(recall $-\infty < t < 0$)



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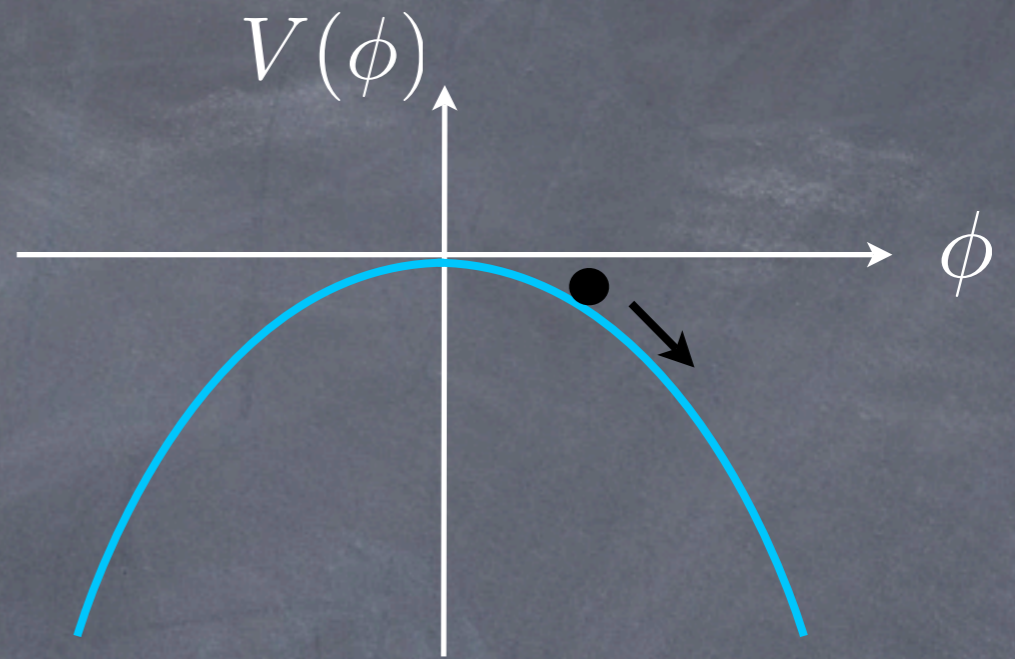
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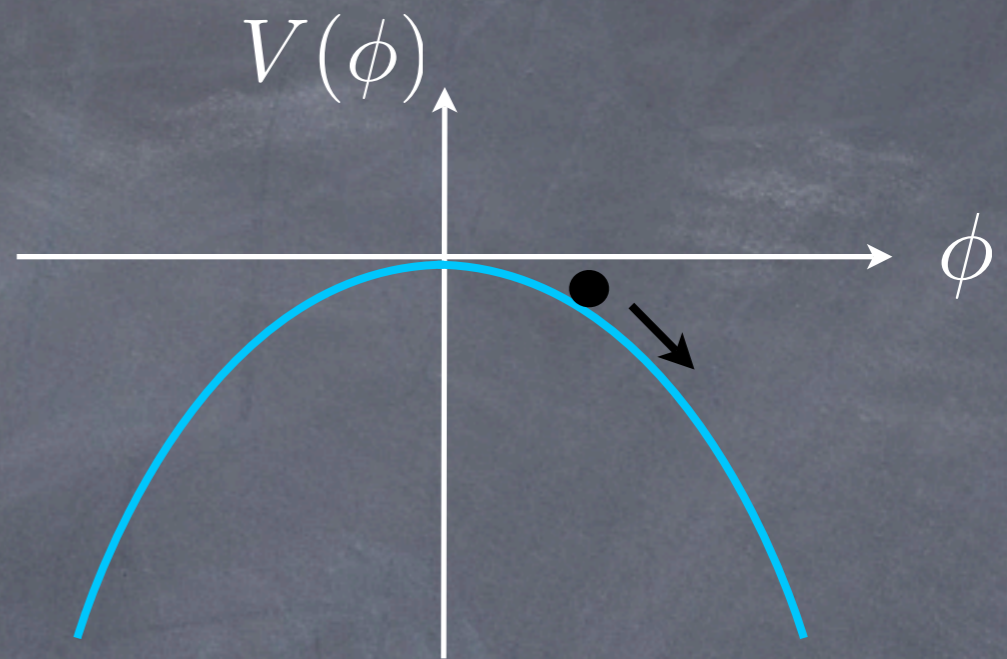
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Quantum fluctuations: assuming adiabatic vacuum,

$$\varphi_k = \frac{\sqrt{-t}}{4\sqrt{2}\pi} H_{5/2}^{(1)}(-kt) \rightarrow \frac{3}{4\pi^{3/2}} \frac{1}{k^{5/2} t^2}$$

\implies **red spectrum**



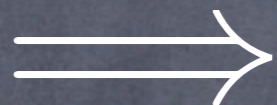
But weight-0 field gets S.I. spectrum:

Assuming $\lambda_\chi = 0$ for simplicity,

$$\ddot{\chi}_k + 2\frac{\dot{\phi}}{\phi}\dot{\chi}_k + k^2\chi_k = 0$$

Choosing adiabatic vacuum,

$$\chi_k = \frac{\sqrt{\lambda}(-t)}{\sqrt{2}} \frac{e^{-ikt}}{\sqrt{2k}} \left(1 - \frac{i}{kt} \right).$$



$$k^{3/2}|\chi_k| \simeq \frac{\sqrt{\lambda}}{2}$$

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- 2nd field only amplifies to a constant (attractor)
- Contrast from other 2-field mechanisms...
- No special tuning necessary

e.g. Buchbinder, Khoury & Ovrut, hep-th/0702154
Creminelli & Senatore, hep-th/0702165
Tolley & Wesley, hep-th/0703101

Existing examples

• Rubakov's U(1) model: $\mathcal{L}_{\text{Rubakov}} = -\frac{1}{2}\partial_\mu\psi\partial^\mu\psi^* + \frac{\lambda}{4}|\psi|^4$
0906:3693

In terms of polar coordinates, $\psi = \phi e^{i\chi}$,

$$\mathcal{L}_{\text{Rubakov}} = -\frac{1}{2}(\partial\phi)^2 + \frac{\lambda}{4}\phi^4 - \frac{1}{2}\phi^2(\partial\chi)^2$$

- ψ is conformally coupled to gravity (here: minimally-coupled)
- ψ is a spectator field (here: drives cosmology)

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- Galilean Genesis [Creminelli, Nicolis & Trincherini, 1007:0027](#)

$$\mathcal{L}_{\text{Genesis}} = \frac{1}{2}(\partial\phi)^2 + c \left(\frac{1}{2} \frac{(\partial\phi)^2 \square\phi}{\phi^3} - \frac{1}{4} \frac{(\partial\phi)^4}{\phi^4} \right).$$

wrong sign $\implies \phi = \frac{\sqrt{3c}}{\sqrt{2}(-t)}$

- Violates Null Energy Condition
- Superluminal propagation

General framework

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On the perturbations, $\varphi_I = \phi_I - \bar{\phi}_I$, we have

$$\begin{aligned} \delta_{P_i} \varphi_I &= -\partial_i \varphi_I, & \delta_{P_0} \varphi_I &= -\frac{d_I}{t} \bar{\phi}_I - \dot{\varphi}_I, \\ \delta_{J^{ij}} \varphi_I &= (x^i \partial^j - x^j \partial^i) \varphi_I, & \delta_{J^{0i}} \varphi_I &= \frac{d_I}{t} x^i \bar{\phi}_I + (t \partial_i + x^i \frac{d}{dt}) \varphi_I, \\ \delta_D \varphi_I &= -(d_I + x^\mu \partial_\mu) \varphi_I, & \delta_{K_0} \varphi_I &= x^2 \frac{d_I}{t} \bar{\phi}_I + \left(2t d_I + 2t x^\nu \partial_\nu + x^2 \frac{d}{dt} \right) \varphi_I. \\ \delta_{K_i} \varphi_I &= (-2x_i d_I - 2x_i x^\nu \partial_\nu + x^2 \partial_i) \varphi_I. \end{aligned}$$

Phenomenological Lagrangian

$$\mathcal{L}_{\text{quad}} = \frac{1}{2} M_1^{IJ}(t) \dot{\varphi}_I \dot{\varphi}_J - \frac{1}{2} M_2^{IJ}(t) \vec{\nabla} \varphi_I \cdot \vec{\nabla} \varphi_J - \frac{1}{2} M_3^{IJ}(t) \varphi_I \varphi_J$$

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Imposing linearly and non-linearly realized symmetries:

$$\mathcal{L}_{\text{quad}}^{(d_I \neq 0)} \sim (-t)^{2(d_I-1)} \eta^{\mu\nu} \partial_\mu \varphi_I \partial_\nu \varphi^I - (-t)^{2(d_I-2)} (d_I + 1)(d_I - 4) \varphi_I \varphi^I$$

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Exception is the $d_I = 0$ sector:

$$\mathcal{L}_{\text{quad}}^{(d_I=0)} \sim t^{-2} \eta^{\mu\nu} \partial_\mu \chi_I \partial_\nu \chi^I - \tilde{M}_3^{IJ} t^{-4} \chi_I \chi_J$$

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In progress...

Hinterbichler, Joyce & Khoury, to appear

- Coset construction: non-linear realization of $SO(4,2)$, with linearly realized $SO(4,1)$ subgroup.

cf. Coleman, Wess & Zumino (1969); Salam & Strathdee (1969); Low & Manohar (2002).

Turning on Gravity

$$S = \int d^4x \sqrt{-g_E} \left(\frac{M_{\text{Pl}}^2}{2} R_E - \frac{1}{2} (\partial\phi)^2 + \frac{\lambda}{4} \phi^4 - \frac{1}{2} \frac{\phi^2}{M^2} g^{\mu\nu} \partial_\mu \tilde{\chi} \partial_\nu \tilde{\chi} \right) .$$

where matter fields $\tilde{\chi} = M\chi$ couple to

$$g_{\mu\nu}^{\text{J}} = \frac{\phi^2}{M^2} g_{\mu\nu}^{\text{E}}$$

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At sufficiently early times, gravity can be neglected:

$$\phi(t) \simeq \frac{\sqrt{2}}{\sqrt{\lambda}(-t)} \quad -\infty < t < 0$$

$$\dot{H}_{\text{E}} = -\frac{1}{2M_{\text{Pl}}^2} \dot{\phi}^2 = -\frac{1}{\lambda t^4 M_{\text{Pl}}^2} \implies H_{\text{E}}(t) \approx \frac{1}{3\lambda t^3 M_{\text{Pl}}^2}$$

Contracting universe

Flatness and Homogeneity

$$H_E(t) \approx \frac{1}{3\lambda t^3 M_{\text{Pl}}^2} \implies$$

$$a_E(t) \simeq 1 - \frac{1}{6\lambda t^2 M_{\text{Pl}}^2}$$

Nearly static

Flatness and Homogeneity

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Nearly static

Universe becomes increasingly flat and homogeneous:

$$3H_E^2 M_{\text{Pl}}^2 = \frac{K}{a_E^2} + \frac{C_{\text{mat}}}{a_E^3} + \frac{C_{\text{rad}}}{a_E^4} + \frac{C_{\text{aniso}}}{a_E^6} + \dots + \rho_\phi$$

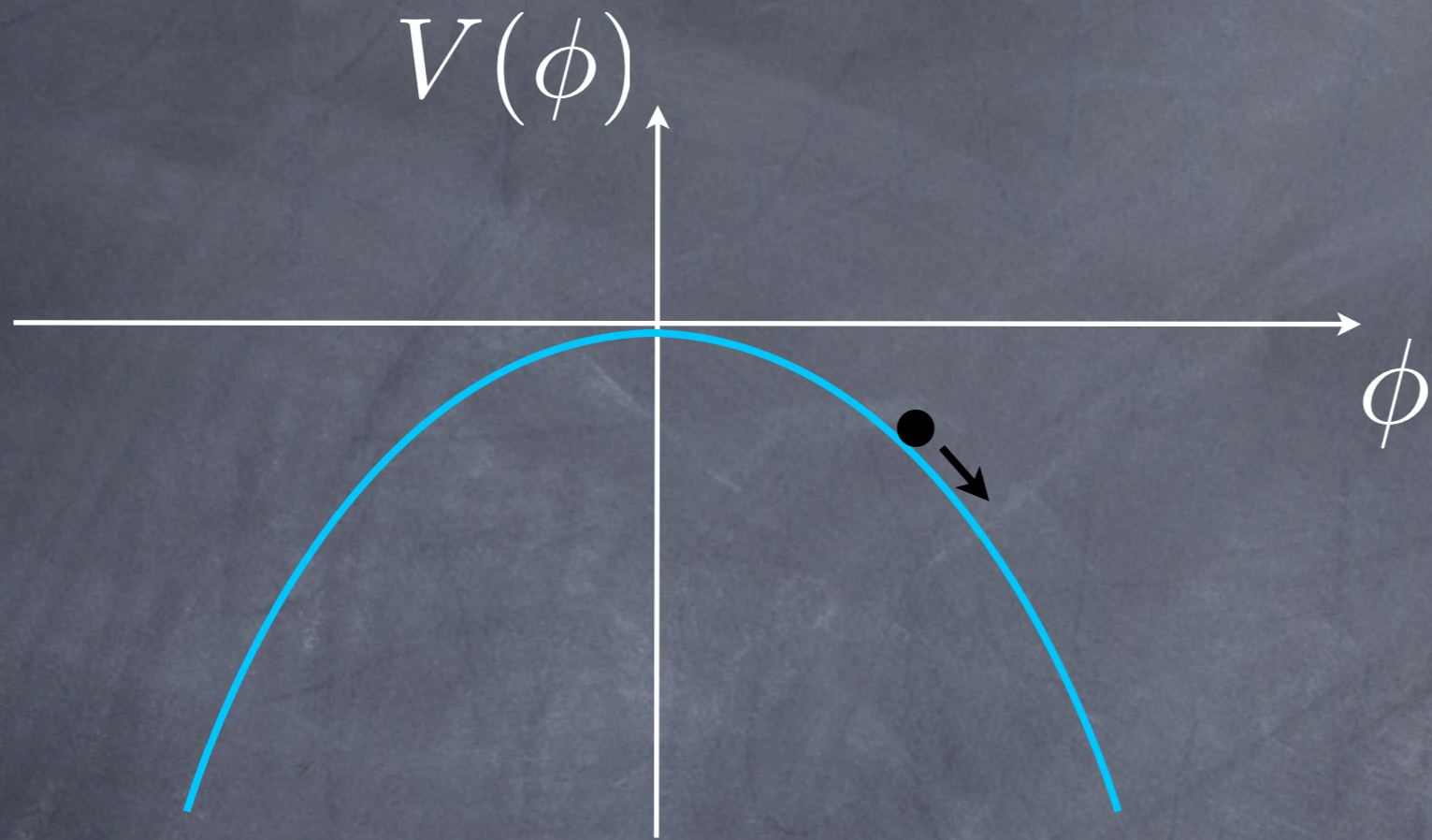
$\simeq \text{const.}$

$\sim \frac{1}{\lambda^2 t^6 M_{\text{Pl}}^2}$

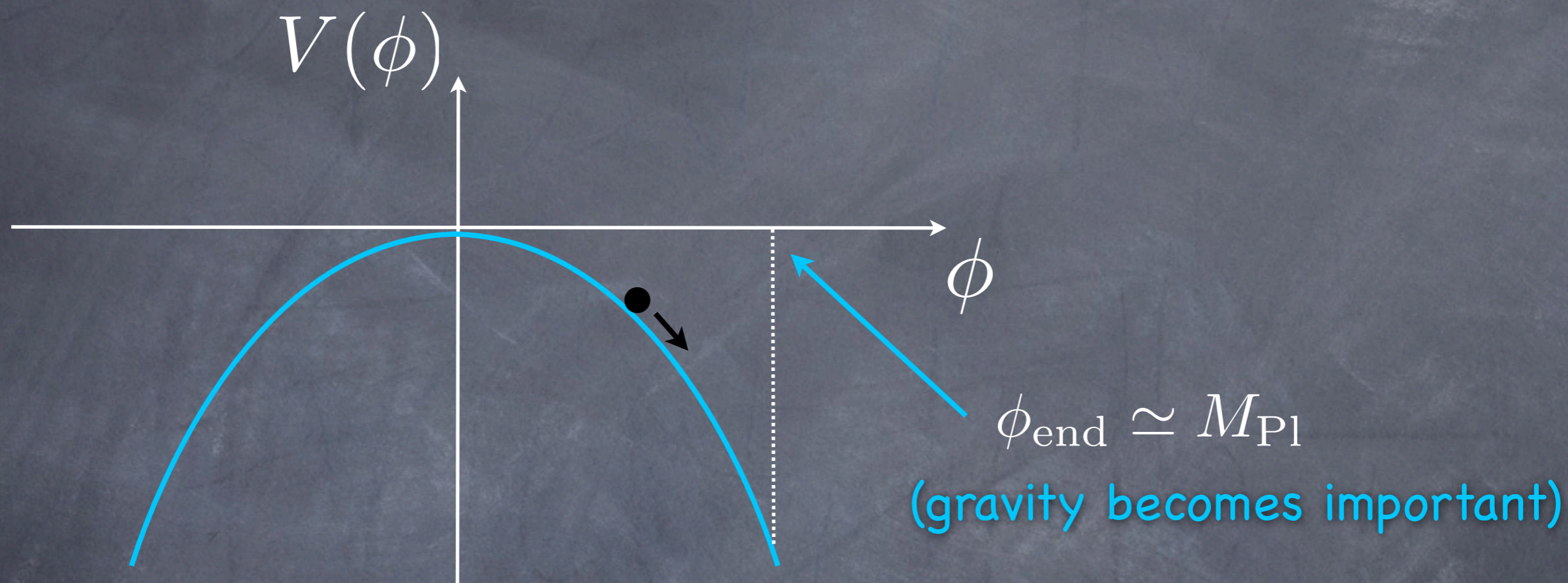
Akin to ekpyrotic cosmologies (contracting universe with $w \gg 1$)

Gratton, Khoury, Steinhardt & Turok (2003);
Erickson, Wesley, Steinhardt & Turok (2004).

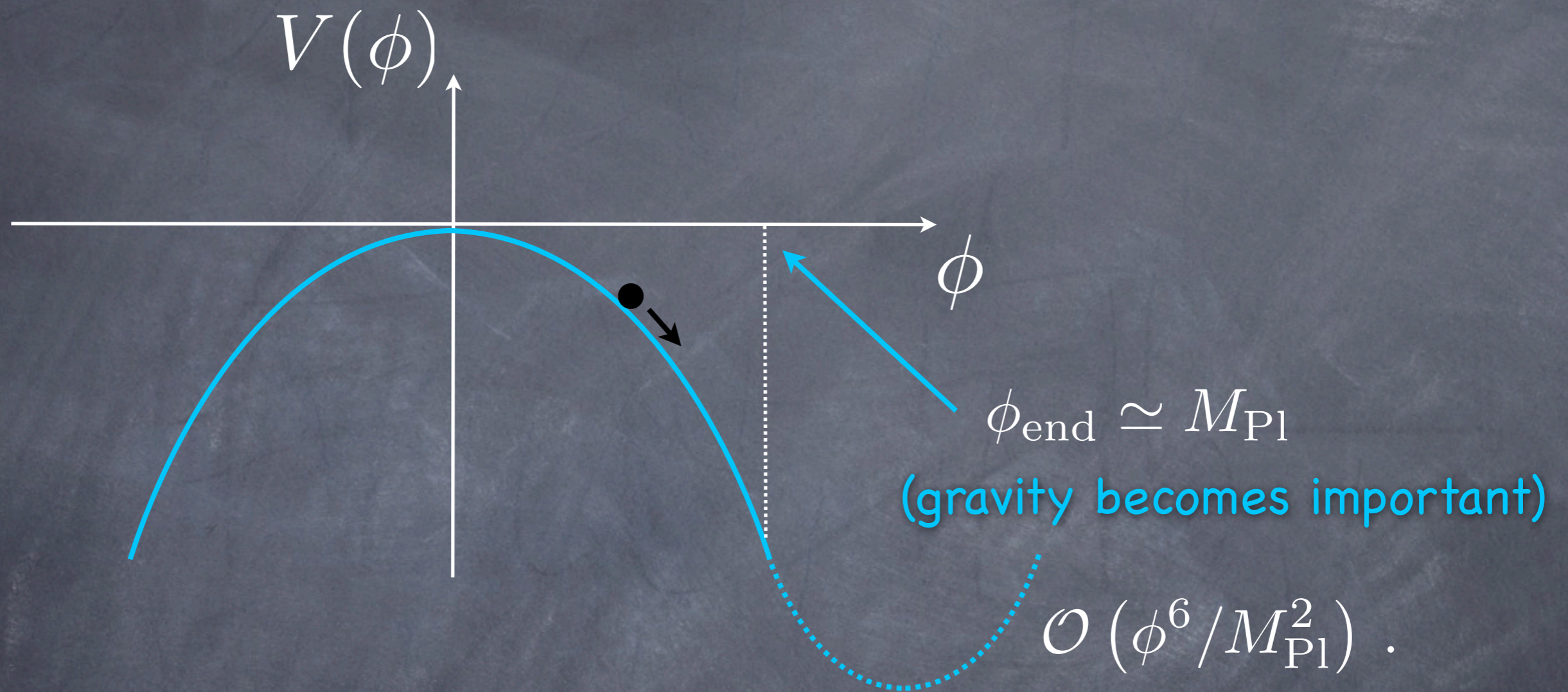
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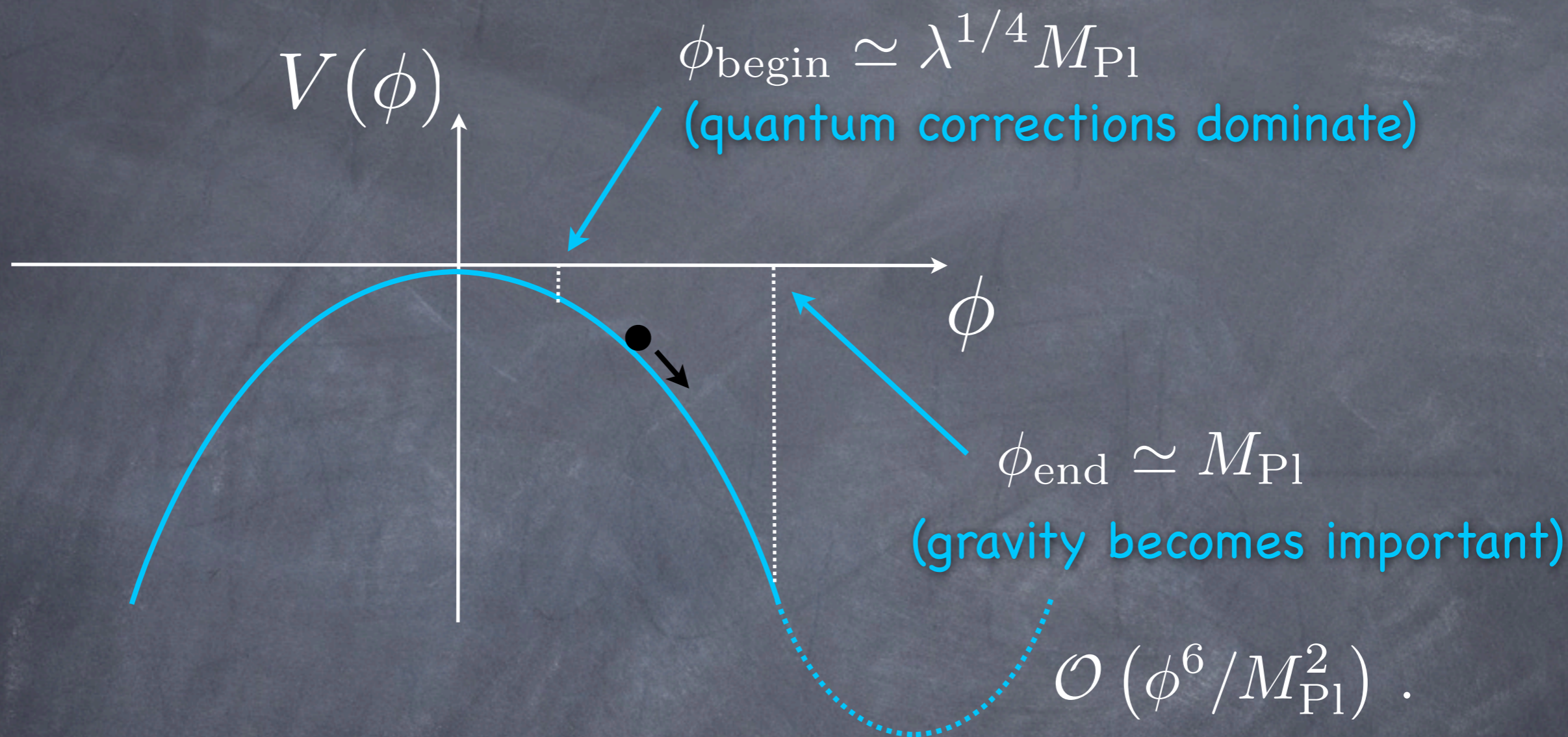
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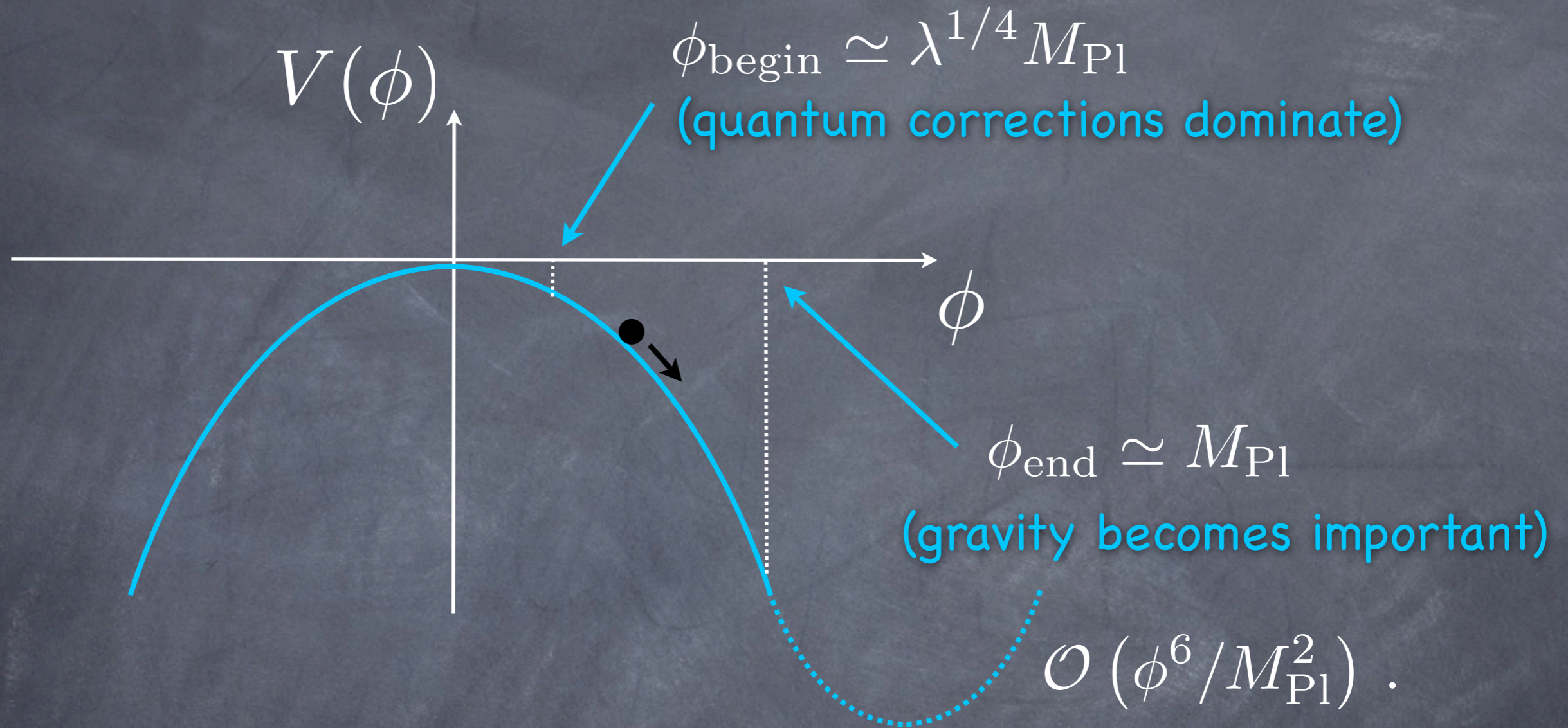
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$$\Rightarrow \frac{k_{\text{end}}}{k_{\text{begin}}} = \frac{\phi_{\text{end}}}{\phi_{\text{begin}}} = \lambda^{-1/4}$$

Note: Modes are all super-Hubble by ϕ_{end}

DBI extension

Hinterbichler, Khoury & Miller, to appear

$$\mathcal{L}_{\text{DBI}} = -\phi^4 \sqrt{1 + \frac{(\partial\phi)^2}{\phi^4}} + \left(1 + \frac{\lambda}{4}\right) \phi^4$$

Special conformal transformations realized non-linearly:

$$\delta_\epsilon \phi = \epsilon(\phi + x^\mu \partial_\mu \phi);$$

$$\delta_v \phi = v_\mu x^\mu \phi - \partial_\mu \phi \left(\frac{1}{2} v^\mu x^2 - (v \cdot x) x^\mu - \frac{1}{2\phi^2} v^\mu \right).$$

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Equation of motion is more intricate...

$$-\frac{d}{dt} \left(\frac{\dot{\phi}}{\sqrt{1 - \dot{\phi}^2/\phi^4}} \right) - \frac{2\dot{\phi}^2}{\phi \sqrt{1 - \dot{\phi}^2/\phi^4}} + 4\phi^3 \left(1 + \frac{\lambda}{4} - \sqrt{1 - \dot{\phi}^2/\phi^4} \right) = 0$$

...but still admits a 1/t solution:

$$\phi(t) = \frac{1 + \lambda/4}{\sqrt{1 + \lambda/8}} \frac{\sqrt{2}}{\sqrt{\lambda}(-t)}$$

Conclusions

- Spontaneous conformal symmetry breaking: $SO(4, 2) \rightarrow SO(4, 1)$

- For scalar ϕ_I of conformal weight d_I , this is realized by

$$\phi_I \sim t^{-d_I}$$

- Action for Goldstones φ_I fixed by symmetry breaking pattern

\implies Background is attractor

\implies Conformal weight-0 fields χ acquire S.I. spectrum

- Gravity unimportant at early times, universe driven to flatness and homogeneity

- Generic Predictions:
 - significant non-gaussianity
 - negligible gravity waves